Introduction

• Goal is to improve knowledge and appreciation of the art of particle accelerator physics
• Accelerator health physicists should understand how the machines work.
• Accelerators have unique operational characteristics of importance to radiation protection.
  – For their own understanding
  – To promote communication with accelerator physicists, operators, and experimenters
• This course will not make you an accelerator PHYSICIST!
  – Limitations of both time and level prescript that.
  – For those who want to learn more, academic courses and the U. S. Particle Accelerator School provide much comprehensive opportunities.
• Much of the material is found in several of the references.
  – Particularly clear or unique descriptions are cited among these.
A word about notation

• Vector notation will be used extensively.
  – Vectors are printed in italic boldface (e.g., $E$)
  – Their corresponding magnitudes are shown in italics (e.g., $E$).

• Variable names generally will follow the published literature.

• Consistency has not been achieved.
  – This author cannot fix that by himself!
  – Chose to remain close to the literature
  – Watch the context!
Summary of relativistic relationships including Maxwell’s equations

- Special theory of relativity is important.
- Accelerators work because of Maxwell’s equations.
- The rest energy of a particle $W_o$ is connected to its rest mass $m_o$ by the speed of light $c$:

$$W_o = m_o c^2 \quad (1)$$

- Total energy $W$ of a particle moving with velocity $v$ is

$$W = mc^2 = \frac{m_o c^2}{\sqrt{1 - \beta^2}} = \gamma m_o c^2$$

$$\gamma = \frac{W}{W_o} = \frac{1}{\sqrt{1 - \beta^2}}, \quad (2)$$

$\beta = v/c$, $m$ is the relativistic mass, and $\gamma$ is the relativistic parameter.
Summary of relativistic relationships

Thus kinetic energy \( T = W - W_o \),

\[
m = \frac{m_o}{\sqrt{1 - \beta^2}} = \gamma m_o \quad (3) \quad \text{and} \quad \beta = \sqrt{1 - \left(\frac{W_o}{W}\right)^2} \quad (4)
\]

Momentum \( p \) is related to relativistic \( m \) and \( v \):

\[
p = mv = \gamma m_o \beta c = \frac{W m_o}{m_o c^2} \left[ \sqrt{1 - \left(\frac{W_o}{W}\right)^2} \right] c = \frac{1}{c} \left[ \sqrt{W^2 - W_o^2} \right] = \frac{1}{c} \sqrt{T(T + 2W_o)} \quad (5)
\]

At high energies \( p \approx T/c \approx W/c \) \ At low energies \( p^2 \approx (2W_o/c^2)T = 2m_o T \)

Also,

\[
W^2 = p^2 c^2 + m_o^2 c^4 \quad (6)
\]
Summary of relativistic relationships

• Convenient to work in a system of units where energy is in units of eV, MeV, etc.
• Velocities are then expressed in units of the speed of light \(0 \leq \beta \leq 1\), momenta as energy divided by \(c\) (e.g., MeV \(c^{-1}\), etc.), and masses as energy divided by \(c^2\) (e.g., MeV \(c^{-2}\), etc.).
• In these so-called “energy” units \(W\) and \(m\) are numerically equivalent. One thus does not need the numerical value of \(c\) or \(c^2\)
• Nearly always, if a particle energy is referred to as being, say, “100 MeV”, the kinetic energy \(T\), not the total energy \(W\), is meant.
  – Is an important distinction at low energies!
Maxwell’s equations

Maxwell’s equations in vacuum in SI units will be used. They connect

the electric field $E$ [Volts (V) m$^{-1}$],
the magnetic field $B$ [Tesla (T)],
the charge density $\rho(r, t)$ [Coulombs (C) m$^{-3}$],
and current density $j(r, t)$ [Amperes (A) m$^{-2}$] (units wrong in text!)
at a location in space $r$ at time $t$.

$\varepsilon_o$ is the dielectric constant in vacuum [$10^7/(4\pi c^2)$ (C V$^{-1}$ m$^{-1}$)]
$\mu_o$ is the permeability of free space [$4\pi \times 10^{-7}$ (V s A$^{-1}$ m$^{-1}$)],
$c^2=1/(\mu_o \varepsilon_o)$. This fact led Einstein to special relativity!
Maxwell’s equations

The equations in differential (left) and integral forms (right) are

\[ \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho \quad \text{or} \quad \oint_{4\pi} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho \, dV \quad \text{(Gauss’s Law), (7)} \]

\[ \nabla \cdot \mathbf{B} = 0 \quad \text{or} \quad \oint_{4\pi} \mathbf{B} \cdot d\mathbf{S} = 0 \quad \text{(8)} \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{or} \quad \oint \mathbf{E} \cdot d\ell = -\oint \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \quad \text{(Faraday’s Law), (9)} \]

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad \text{or} \quad \oint \mathbf{B} \cdot d\ell = \mu_0 \oint \mathbf{j} \cdot d\mathbf{S} + \frac{1}{c^2} \oint \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S} \quad \text{(Ampere’s Law), (10)} \]
Deflection of Charged Particles

Electromagnetic forces are used to accelerate, deflect, and focus charged particles.

The Lorentz force $F$ [Newtons (N)] on charge $q$ (C) with velocity $v$ (m s$^{-1}$) due to applied electric field $E$ (V m$^{-1}$) and magnetic field $B$ (T) is

$$F = q(v \times B + E) = \frac{dp}{dt},$$  \hspace{1cm} (11)

where $p$ is momentum in SI units and $t$ is the time (s)

Static electric fields (i.e., $dE/dt=0$) accelerate or decelerate charged particles.

Static magnetic fields $(dB/dt=0)$ can only deflect them.

Nonstatic conditions couple these fields via Eqs. (9) and (10).
Some considerations

• Nature and present-day technology have placed constraints on the design of static deflection and focusing devices.
  – Static electric fields are limited a few MV m\(^{-1}\).
  – Ferromagnetic materials are limited to about 2 T and superconductors to 4-10 T.

• A common simplification is that the physical boundaries of a given deflecting or focusing element exactly define the boundary of the electric or magnetic field.
  – For real components the fields extend beyond the geometrical boundaries for short distances in fringe fields.
  – Fringe fields also deflect the particles.
  – The beam elements behave as if they are a bit larger than their geometric profiles.
  – Thus the concepts of an effective boundary and an effective field integral are used.
A digression on synchrotron radiation

- Accelerated charged particles radiate energy, even if the acceleration is purely centripetal.
- Thus particles even in a circular orbit lose energy by radiating it as photons.
- This is the basis for synchrotron radiation, or “light” sources.
- It also impacts accelerator physics
  - Has to be “fed” by RF energy
  - There are other subtle effects on beam properties.
- For highly relativistic energies, readily achieved for electrons, the photons emerge in a tight bundle along a tangent to any point on a circular orbit.
A digression on synchrotron radiation

- The characteristic angle $\theta_c$ is defined to be the angle where the intensity is $1/e$ (1/2.718) of that at the center.

\[
2\theta_c = \frac{2}{\gamma} = 2\sqrt{1 - \beta^2}
\]

\[
\theta_c = \frac{1}{\gamma} = \sqrt{1 - \beta^2} \quad (12)
\]
A digression on synchrotron radiation

• The power spectrum has a universal shape:

\[ \varepsilon_c = \frac{2.218 W^3}{R} \text{ (keV)} \] (13)

with median energy for electrons for \( W \) (GeV) & bending radius \( R \) (meters). For singly-charged particles of other masses \( m_x \) multiply by \((m_e/m_x)^3\).
A digression on synchrotron radiation

- The radiated power $P$ (watts) for circulating electron current $I$ (ma) is

$$P = \frac{88.46W^4I}{R} \quad (14)$$

For singly charged particles of other masses $m_x$ multiply by $(m/e/m_x)^4$.

- The LHC at CERN will be the first proton accelerator where synchrotron radiation is of crucial importance
- We see why synchrotron radiation sources all use electrons!
Deflection of Charged Particles by Static Electric Fields

Fig. 1 shows purely electrostatic deflection \((dE/dt=0 \text{ and } B=0)\) of a fast particle with initial momentum \(p=p_z\) and velocity \(v\).

Consider a particle of charge \(q\) passing through plates of length \(L\) biased at voltage \(V\), & separated by distance \(\delta\) (all SI units):

\[
\begin{align*}
\text{Fig. 1: Purely electrostatic deflection with } \frac{dE}{dt} = 0 \text{ and } B = 0. \\
\text{Initial momentum } p = p_z \text{ and velocity } v. \\
\text{Consider a particle of charge } q \text{ passing through plates of length } L \text{ biased at voltage } V, \& \text{ separated by distance } \delta. \\
\text{(all SI units)}.
\end{align*}
\]
Pure electrostatic deflection

- From Eq. (7) the electric field $E$ has a uniform value of $V/\delta$ oriented as shown.
- Initially, the components of the momentum are $p_z = p$ and $p_x = p_y = 0$.
- Since $B = 0$, Eq. (11) gives
  \[
  \frac{dp_x}{dt} = qE_x = q \frac{V}{\delta} \tag{15}
  \]
- With no $z$-component of force, no change in $p_z$ occurs in this system.
**Pure electrostatic deflection**

For a “fast” particle, we ignore the small deviation from a “straight” path.

We can integrate over the time $t$ needed to travel through the device and get $p_x$ at the exit:

$$p_x(\text{exit}) = q \frac{V L}{\delta v} = q \frac{V L}{\delta \beta c} \quad (\text{kg m s}^{-1}) \quad (16a)$$

For $q$ in units of electron charge this is already in eV/c.

$P_x$ in MeV/c is useful:

$$p_x(\text{exit}) = q \frac{V L}{\delta \beta} \times 10^{-6} \quad (16b)$$
Pure electrostatic deflection

Look at the angular deflection $\Delta \theta$ (radians) and apply the small angle approximation:

$$\Delta \theta = \tan^{-1} \left[ \frac{p_x}{p_z} \right] = \tan^{-1} \left[ \frac{qVL}{p_z \delta \beta} \times 10^{-6} \right] = \tan^{-1} \left[ \frac{qEL}{p_z \beta} \times 10^{-6} \right]$$

for $p_z \gg p_x$

$$\Delta \theta \approx \frac{qEL}{p \beta} \times 10^{-6} \quad (17)$$
Uses of pure electrostatic deflection

- Used as inflectors (injection) and electrostatic septa (extraction).
- At low energies conductor planes are used.
- At high energies, planes of fine wires, as small as 50 μm diameter, commonly approximate one or both of the parallel plates.
- These devices can be tens of meters long, have $V > 100$ kV.
- One conductive plane slices through the beam to split it by giving part of it a small momentum kick.
- Wires are used to minimize material struck by the beam.
- Often used in conjunction with magnetic elements.
Electrostatic deflection radiation protection considerations

• Especially for electrons, synchrotron radiation can result.
• The deflection has to achieve the correct angle.
• Wire septa can create locally strong electric fields
  – Capable of ionizing residual gas atoms
  – Accelerate electrons across the gap as a dark current
  – Produce x-rays [Example: At Fermilab have seen > 1 mGy h⁻¹]
  – This phenomena is not very reproducible (cleanliness, residual pressure = “bad” vacuum, etc.)
• Activation of septa components by stray beam
• Terminology and jargon: Small angles measured in milliradians, readily becomes “mrad”, or even “mr” but absorbed dose is NOT being discussed.
Deflection of Charged Particles by Static Magnetic Fields

Fig. 2 shows a purely magnetostatic deflection with no electric field ($dB/dt=0$, $E=0$).

Eq. (11): $F = \frac{dp}{dt} = q(v \times B + E) = q(v \times B)$

$B$ is perpendicular to the paper and directed toward the reader
Deflection of Charged Particles by Static Magnetic Fields

• Due to the cross product, any component of \( p \) which is parallel to \( B \) will not be altered by it.
• Typically charged particles are deflected by dipole magnets with fields:
  – Often nearly spatially uniform
  – Constant or slowly-varying compared with the time the particle is present.
• If
  – There is no component of \( p \) that is parallel to \( B \)
  – Energy loss from synchrotron radiation can be ignored
• Then
  – The magnetic force supplies the needed centripetal acceleration
  – The kinetic energy of the particle unchanged.
  – A circular path is followed (as shown in Fig. 2)
• A component of \( p \) parallel to \( B \) results in a spiral trajectory with that component conserved.
Deflection of Charged Particles by Static Magnetic Fields

Equating the centripetal force to the magnetic force for $p$ perpendicular to $B$ gives

$$F = q(v \times B) \text{ simplifies to } \frac{mv^2}{R} = qvB \quad (18)$$

where $m$ is the relativistic mass and $p=mv$.

Solving for $R$ and making a convenient unit conversion:

$$R(\text{meters}) = \frac{p \text{ (SI units)}}{qB} = \frac{p(\text{MeV c}^{-1})}{299.79qB} \quad (19)$$

In the far right-hand-side, $q$ is now in # of electronic charges,

$B$ remains in Tesla

The “300” factor is the speed of light in SI units divided by $10^6$.

The useful product $BR$ (T-m) is called the magnetic rigidity; Greek “$\rho$” is often used instead of $R$, so that the quantity is called “B-rho”.
Deflection of Charged Particles by Static Magnetic Fields

Look at Fig. 2 again

For short magnets $L \ll R$, the deflection $\Delta \theta$ for momentum $p$ (MeV c$^{-1}$) is given by

$$\Delta \theta = \frac{L}{R} \frac{299.79qBL}{p} \text{ (radians)} \quad (20)$$

$BL$ (T m) is called the field integral of the magnet system.
Magnetic deflection radiation protection considerations

• Need to make calculations to miss objects, clear apertures, etc.
• Can use this to select charge states or momenta
• Used to measure momenta, can be needed to assure safe design of facilities downstream
• Part of radiation safety system design
  – Electro-mechanically, interlocks must be designed to high standards
  – The design in terms of beam physics; to preclude unwanted charged particles, is just as important
• Usually, magnet current “off” is the easiest “safe” configuration.
Example of Bending Magnets

Fermilab 120 GeV Main Injector Bending Magnets

during assembly

as assembled

Photos from Fermilab website
Focusing of Charged Particles

• Dominant technique: quadrupoles with “near static” fields
  – Electric quadrupoles
  – Magnetic quadrupoles

• Higher order multipoles; sextupoles, octupoles, etc. are used, beyond the scope of this course
  – Correct for imperfections from “ideal”

• Will also discuss other focusing techniques
  – Principally adaptations of dipole magnets
Quadrupole lenses

- Will use Cartesian coordinates & “right-hand” rule
- The $z$-axis, the optic axis, is into the screen
- Values of $x > 0$ are “beam’s eye” left
- Values of $y > 0$ are “beam’s eye” up
- Below types are set to focus + charge the same way
- Orientations differ due to Eq. (11) cross product
- Electric and magnetic field lines are shown between the poles
- Gap radius $a$ is defined.
- The length of the
- quadrupole is $L$
- See Fig. 3:
Quadrupole lenses

“Ideal” pole pieces (equipotentials) are hyperbolae;

- Electric: \(x^2 - y^2 = \pm a\)
- Magnetic: \(xy = \pm a^2/2\)
Quadrupole lenses

$B_o \ (T)$ is magnetic field at magnetic pole pieces

$E_o = 2V/a \ (V \ m^{-1})$ is electric field at electric pole pieces

For magnetic quad, the field components in the gap are:

$$B_x = -\frac{B_o}{a}y = -g_m y \quad (21) \quad B_y = -\frac{B_o}{a}x = -g_m x \quad (22)$$

For electric quad, use same Eqns. with $E_o$ replacing $B_o$

The gradients, $g_m \ (T \ m^{-1})$ and $g_e \ (V \ m^{-2})$ are defined.
Quadrupole lenses-qualitative deflections
(See p. 11 of chapter) Need >1 to focus in both planes!
Quadrupole lenses—let’s add math

In the $yz$-plane of a magnetic quadrupole, apply Eq. (20):

$$\Delta \theta \approx \frac{L}{R} = \frac{299.79 qBL}{p} \text{ (radians)} \quad (20)$$

Substitute $B_x = -g_m y$, and get the angular deflection $\Delta \theta$:

$$\Delta \theta = \frac{299.79 q g_m y L}{p} \text{ (radians)} \quad (23)$$

For electric quadrupole, apply Eq. (17):

$$\Delta \theta \approx \frac{qEL}{p \beta} \times 10^{-6} \text{ (radians)} \quad (17)$$

And get

$$\Delta \theta = \frac{q L g_y}{p \beta} \times 10^{-6} \text{ (radians)} \quad (24)$$

Note: For both $\Delta \theta$ is proportional to $y$, a feature of focusing of visible light by lenses.

For awhile, will treat electric and magnetic quads the same.
**Quadrupole lenses- focusing action**

Start with particles traveling parallel to the $z$-axis displaced by $y$. After deflection by a small angle $\Delta \theta$, will intercept the $z$-axis at distance $f$:

$$ f = \frac{y}{\tan \Delta \theta} \approx \frac{y}{\Delta \theta} = \frac{p}{299.79 q L g_m} \quad \text{(magnetic)} $$

or

$$ f \approx \frac{y}{\Delta \theta} = \frac{p \beta}{q L g_e} \times 10^6 \quad \text{(electric) (25)} $$

Note that $f$ is independent of $y$! This is the “thin lens” approximation, true if $f \gg L$. $f$ is, naturally, the focal length. See Fig. 4a:
**Quadrupole lenses-thin lens equation**

The analogy with “light” optics is true mathematically, we have the thin lens equation connecting the image distance $z_i$, with the object distance $z_o$;

$$\frac{1}{z_o} + \frac{1}{z_i} = \frac{1}{f} \quad (26)$$

If $f > 0$ with $z_o > 0$ and $z_i > 0$, get a real image in the focusing plane.

In the defocusing plane, $f < 0$, get a virtual image since for $z_o > 0$, $z_i < 0$.

If lenses are thick, reference principal planes rather than center of the quadrupole (more later!) See Fig. 4b:
**Quadrupole doublets**

Need multiple elements to focus in BOTH planes
Quadrupole doublets (pair of 2) is the most common solution
See Fig. 4c (*yz-plane*)
and
Fig. 4d (*xz-plane*)

Will assume
quad 1 and
quad 2 have
different focal
lengths. (Often
quads are set
up identical!)
**Quadrupole doublets**

In this setup, for incoming beam parallel to the $z$-axis in the $yz$-plane, relative to quad 1, $z_{yo1} = \infty$ and $z_{yi1} = f_1$.

The quad 2 object distance, a virtual one, is $z_{yo2} = d - f_1$.

Relative to quad 2, the image distance is obtained from:

$$\frac{1}{z_{yi2}} = \frac{1}{-f_2} - \frac{1}{d - f_1} \quad (27)$$

($f_2 < 0$ implies quad 2 defocuses)

Solving:

$$z_{yi2} = \frac{f_2(f_1 - d)}{f_2 - f_1 + d} \quad \text{or} \quad z_{yi2} = \frac{f(f - d)}{d} \quad \text{for identical quadrupoles} \quad (28)$$
**Quadrupole doublets**

Likewise, for the $xz$-plane, the final image distance $z_{xi2}$ is given by

$$z_{xi2} = \frac{f_2(f_1 + d)}{f_1 - f_2 + d} \quad \text{or} \quad z_{xi2} = \frac{f(f + d)}{d} \quad \text{for identical quadrupoles (29)}$$

Note that for identical quadrupoles,

$$z_{xi2} - z_{yi2} = 2f \quad (30)$$

Expected! This focused in $yz$ before $xz$. (Not identical to light optics!)

Average focal length is $f^2/d$. 
**Quadrupole triplets**

- More complicated systems overcome some limitations
- A commonly used, elegant variant is that of the quadrupole triplet
  - 3 quadrupoles, quad 1, quad 2, and quad 3
  - Alternate in polarity, quad 1 same as quad 3 opposite quad 2
  - Acts like a pair of quadrupole doublets placed back to back
  - Saves space, less distortion, acts (mostly!) like a single lens

- Useful special case:
  - Quad 1 and 3 have equal focal lengths $f$
  - Quad 2 has focal length $\frac{1}{2}f$ (i.e., fields are 2x as strong or magnet is 2x as long)
  - Separation distance is $d$ between each pair.

- Effective focal length $f^*$ for $f >> d$ is

$$f^* = f^2 \left[ 2d \left(1 + \frac{d}{f}\right) \right]^{-1} \approx \frac{f^2}{2d} \quad (31)$$
Examples of magnetic quadrupoles

Indiana University Cyclotron Facility (IUCF)  Fermilab Main Injector

Photo from IUCF website  Photo from Fermilab website
Focusing Mechanisms by Dipole Magnets

- A variety of focusing mechanism using dipole magnets are possible
- Will discuss here several examples
- There are many applications that use combinations of these techniques, even within the same magnet
- Often used in combination with both kinds of quadrupoles, other devices
- Often necessary for proper functioning of accelerators
- Uses:
  - To focus beams on experimental targets
  - To separate particles by energy, mass, or momenta
    - Magnetic spectrometers to separate particles by momenta
    - Mass analyzers
    - To select beams for further acceleration
  - In accelerators to assure desired beam properties
**Sector magnets**

- Used to both bend and focus beams
- Fig. 5 shows example
  - Defines coordinate system, central ray, & median plane
- Generalized, non-uniform $B$ field
Sector magnets

- Field non-uniformity characterized by field index $n$, commonly used at accelerators, used even for edges of uniform field magnets; if $B$ decreases with increasing $r$, $n > 0$.

$$B_z = B_o \left( \frac{r}{R} \right)^{-n} = \frac{\text{constant}}{r^n}; \quad (32a) \text{ equivalently, } n = \frac{-dB}{B} \frac{dr}{r} \quad (32b)$$
Sector magnets; uniform field case

- Uniform field implies by \( n=0 \)
- Has no radial component and hence no first order focusing action
- Still get geometric focusing
  - Trajectories with \( r < R \) travel shorter distance in the field than trajectories with \( r \geq R \), deflected less
  - Have Barber's Rule (see Fig. 6):
    
    If \( \alpha < 180 \) degrees, \( \alpha + \gamma_1 + \gamma_2 = 2\pi \) radians (180 degrees)
Sector magnets; non-uniform fields

Can use a Taylor expansion and the definition of the field index $n$ to get a radial component of the field $B_r(z)$; above and below the midplane.

$$B_r(z) = B_{r,0} + \frac{z}{1!} \left[ \frac{\partial}{\partial z} B_r(z = 0) \right] + \frac{z^2}{2!} \left[ \frac{\partial^2}{\partial z^2} B_r(z = 0) \right] + \ldots (33)$$

From symmetry, the 1st and all even-numbered terms vanish.
**Sector magnets; non-uniform fields**

With no electric currents or fields in the magnet gaps (the usual case!), Eq. (10):
\[ \nabla \times B = \mu_0 j(r, t) + \frac{1}{c^2} \frac{\partial E}{\partial t}, \text{ here } \nabla \times B = 0 \]
and Eq. (32b)
\[ n = \frac{-dB}{B} \frac{dr}{r} \text{ (32b) or } \frac{dB}{dr} = -n \frac{B}{r} \]
leads to:
\[ \frac{\partial B_r(z = 0)}{\partial z} = \frac{\partial B_z(z = 0)}{\partial r} = -\frac{nB_o}{R} \text{ (34)} \]
then
\[ B_r \approx \frac{-nB_o z}{R} \text{ (35)} \]
Sector magnets: non-uniform fields

The relationship between $B_r$ and $z$ is linear!
This linearity leads to focusing just as with quadrupoles!
But, only get such focusing in both horizontal and vertical planes
simultaneously if $0 < n < 1$
Magnets with $n = \frac{1}{2}$ are a VERY special case

Here the focal length $f$ in both planes is the same, given by:

$$\frac{1}{f} = \frac{1}{R\sqrt{2}} \sin \left( \frac{L}{R\sqrt{2}} \right)$$  (36)

$L$ is the central ray path length through the magnet
$f$ is measured from the magnet’s effective boundary, a principal plane
Eq. (36) $\Rightarrow f > R$, system length $> 4R$. 

$$B_r \approx -\frac{nB_0z}{R}$$  (35)
**Sector magnets; weak and strong focusing**

- **Limitation:** $f$ gets very large at high energies because $R$ gets large!
- $n < 1$ is called **weak focusing**
  - Used in cyclotrons, betatrons, synchrocyclotrons and 1st generation synchrotrons (e.g., Bevatron, Cosmotron, ANL-ZGS era machines)
- $n >> 1$ is called **strong focusing**
  - Used in 2nd generation synchrotrons (e.g., Cornell, BNL-AGS, CERN-PS)
  - Magnets “alternated” in sense of focusing to achieve overall focusing conditions. Hence, the term “alternating gradient”!

AGS magnet
Diagrams from BNL website

Cosmotron (3 GeV) and AGS (30 GeV) magnets
Sector magnets; weak and strong focusing

- Alternating gradient magnets are expensive to make
- Later synchrotrons (e.g., FNAL, CERN SPS, most light sources) use uniform field dipoles with quadrupoles interleaved in a separated function configuration to achieve strong focusing
- Separates the focusing and bending functions
- Uniform field dipoles \((n = 0)\) are cheaper to make
- Magnets at large accelerators ignore the curvature of the ring, are simply rectangles, a further simplification.
**Entrance and exit edge focusing by uniform field dipoles**

- The edges or boundaries in uniform field dipoles \((n=0)\) can give focusing action.
- Get some radial focusing in edge fields.
- Works if pole pieces are not perpendicular to the beam optic access, rather incident at angle \(\psi\).
- See Fig. 7 for horizontal plane views and coordinate system.
- For \(\psi > 0\) and \(x > 0\), added path length through field gives additional deflection angle:
  \[
  \alpha_h = \frac{x \tan \psi}{R} \quad \text{(37)}
  \]
  linearity proportional to \(x\)!
**Edge focusing-horizontal (bend) plane**

For \( \psi > 0 \) and \( x > 0 \), added path length through field gives additional deflection angle: linearly proportional to \( x \!\)!

\[
\alpha_h = \frac{x \tan \psi}{R} \tag{37}
\]

This gives a focal length of \( R / \tan \psi \) measured from the principal plane of the effective magnet boundary.

Effect limited at high energy accelerators (focal length grows with \( R \!\)!)  
Clearly defocuses in this plane for \( \psi < 0 \).
**Edge focusing-vertical plane**

- What happens in the vertical plane? See Fig. 8.
- Need to calculate the bending effect of fringe field.
- Use Eq. (10)

\[ \oint B \cdot d\ell = \mu_0 \int_S j \cdot dS + \frac{1}{c^2} \int_S \frac{\partial E}{\partial t} \cdot dS = 0 \] (no currents or \( E \) fields!)

\[ \oint B \cdot d\ell = B_0 z + \int_{\text{fringe}} B_s ds = 0 \quad (38) \]
**Edge focusing-vertical plane**

\[ \oint B \cdot d\ell = B_0 z + \int_{\text{fringe}} B_s ds = 0 \quad (38) \]

is true because on our integration loop

- \( B = 0 \) on the left leg (go out far enough!)
- \( B_s = 0 \) in the median plane due to symmetry (bottom leg)

Can thus get field integral of \( B_s \).

But, \( B_s \) is only one component of the total component of the fringe field that is perpendicular to the pole piece.
**Edge focusing-vertical plane**

The vertical deflection sought \( \alpha_v \) is due to the other component of the fringe field; \( B_x = B_s \tan \psi \),

\[
\alpha_v = \frac{1}{RB_o} \int_{\text{fringe}} B_x \, ds = \frac{\tan \psi}{RB_o} \int_{\text{fringe}} B_s \, ds = -\frac{\tan \psi}{R} z \quad (39)
\]

The corresponding focal length, again measured from the effective boundary of the magnet is \((-R/\tan \psi)\).

The negative sign reflects opposite sense of the focusing for \( \psi < 0 \).

Systems of such magnets have been used to achieve overall focusing.
Focusing features in combination

Quadrupole-Dipole-Dipole Multiple (QDDM) Spectrometer at the Indiana University Cyclotron Facility

Photo by J. D. Cossairt, Diagram from IUCF Report No.1-73
Radiation protection considerations related to beam focusing

• Effects ignored up to this point
  – Dispersion
    ✓ No beam is perfectly monoenergetic
    ✓ Magnets will separate momentum components like the rainbow produced of “white” light by a glass prism due to different $R$-values
    ✓ Common to use dispersion matching to remove unwanted effects.
    ✓ Example: $n=1/2$ magnets at the former 83 inch Univ. of Michigan cyclotron. (Drawing from J. D. Cossairt, Ph. D. thesis)
Radiation protection considerations related to beam focusing

• Effects ignored up to this point
  – aberrations, specifically chromatic aberration
    (beam optic focal lengths are obviously momentum-dependent!)
  – transverse emittance
    ➢ Product of angular divergence and transverse size
    ➢ Typical units are $\pi$ mm-milliradian
    ➢ In an extracted beam transverse emittance is constant unless increased by scattering, space charge effects, etc.
    ➢ Cannot be decreased in an extracted beam
      ✓ Making the beam smaller in transverse size will increase angular size
      ✓ Shrinking the angular size (i.e., make beam more parallel), will increase the transverse size
Radiation protection considerations related to beam focusing

• Above discussion based on beam being centered on the optic axis
• If beam is off-axis, other effects can happen
  – Usually undesirable
  – With quadrupoles, the magnets will act like dipoles and deflect (bend) the entire beam instead of focus
    ✓ Will bend according to the field integral seen by the particles
    ✓ This is called beam steering
    ✓ Usually results in undesired beam loss.
Present Accelerator Technology
Accelerators; Cockcroft-Walton Generators

- Most elementary accelerator type
- Used for first artificial nuclear reaction: $p + ^7\text{Li} \rightarrow ^4\text{He} + ^4\text{He}$
- Relies in elevating a terminal to high voltage
- Uses high voltage rectifier “ladder” circuit – See Fig. 9.
- At terminal that contains the ion source, a voltage $V$ is achieved.
- Particles of charge $q$ are accelerated to $T = qV$ (eV) (40).

![Diagram of Cockcroft-Walton Generator](image)
Accelerators; Cockcroft-Walton generators

- **Limitations**
  - Ion source has to sit at high voltage (fiber-optics now help with controls)
  - Usually device is located in a large metal-walled room
  - “Sparking” limits $V$ to about 1 MV, 750 kV is typical
- **Beam currents in the mA range are possible**
- **Good duty factor, beam is on 100 % of the time**
- **Used as injectors to other machines**
- **Only used for ions, not electrons**
- **Being replaced by other types of machines**
Electrostatic accelerators; Cockcroft-Waltons

Fermilab 750 keV Cockcroft-Walton H- Source

Photo from Fermilab website
Electrostatic accelerators; Van de Graaffs

- Van de Graaff accelerators have been workhorses in research, both “pure” and applied
- Now used in some industrial processes
- Used in both single-stage and tandem configurations
- See Fig. 10, will discuss single-stage first
Electrostatic accelerators; Van de Graaffs

- Tandem configuration solves some problems
- See Fig. 10
**Electrostatic accelerators; Van de Graaffs**

- 1\textsuperscript{st} stage accelerates by $T_1 = q_1 V$
- 2\textsuperscript{nd} stage accelerates by $T_2 = q_2 V$
- Final energy is $T = (q_1 + q_2) V$ (41)
- Advantages
  - Ion source is at ground potential
  - Uses HV twice
  - Very stable, “easy” to adjust energy by changing $V$
  - 100 per cent duty factor
  - $V$ up to 25 MV achieved
  - mA beam currents
  - Ions span periodic table

*Note: Image of R. Van de Graaff, photo from BNL website.*
Electrostatic accelerators; Van de Graaffs

• Drawbacks
  – Still have the big tank of SF$_6$ or other insulating gas
  – Available voltage $V$
  – Belt dust used to be a big problem – improvement is to use metal pellets in nylon chains instead of belts – pellatrons!

• Confusing terminology; particle microamperes, (pμA)!
  – Used because of charge states, e.g. $^6$Li$^3^+$
  – One electrical microampere (eμA) is $6.25 \times 10^{12}$ particles s$^{-1}$ for singly-charged particles.
  – One particle microampere is defined as a beam current of $6.25 \times 10^{12}$ particles s$^{-1}$
  – Thus, it only takes $(6.25 \times 10^{12} \text{ particles s}^{-1})/3 = 2.08 \times 10^{12}$ particles s$^{-1}$, 1/3 of a particle microampere, of these ions, to get one electrical microampere of electric current.
Radiation protection considerations of electrostatic accelerators

- HV can result in dark current
  - Can accelerate electrons and produce x-rays
- Conditioning can be a problem
- Sparking arises as an issue
- The 100 per cent duty factors can affect responses of radiation safety instruments, usually favorably.
- Single pass machines
  - Implies that all beam can be lost unless demonstrated to be otherwise.
- Limitations lead one toward use of electromagnetic waves
Radiation protection considerations of electrostatic accelerators

The Yale University Tandem Van De Graaff

Photo from Yale University Wright Nuclear Structure Laboratory website
**Electromagnetic waves**

- Used to overcome limitations of potential drop machines
- Now used also in ion sources (not covered here)

In most simple form, have electric field $E(t)$:

$$E(t) = E_o \cos(\omega t + \xi), \text{ with } \omega = 2\pi f$$  \hspace{1cm} (42)

where $E_o$ is the amplitude,

$\xi$ is the phase angle (arbitrary)

$\omega$ (radians s$^{-1}$) is the angular frequency

$f$ (cycles s$^{-1}$, Hz) is the radio-frequency (RF)
Electromagnetic waves

- Magnetic field $B(t)$ has the same form
- Together $E(t)$ and $B(t)$ satisfy Maxwell’s Equations
- Vector flow of energy in free space is the Poynting Vector
  \[ S(t) = \frac{1}{\mu_0} E(t) \times B(t) \] (43)
- Waves in free space travel at speed $c$, have wavelength $\lambda$;
  \[ \lambda = \frac{c}{f} = \frac{2\pi c}{\omega} \] (44)
- No set of e-m waves is truly monochromatic, even for lasers
  - Small spread of frequencies $\Delta \omega$ about a mean $\omega_o$
  - Wave pattern travels with a given group velocity, slightly $< c$
  - Individual oscillations travel with a phase velocity, slightly $> c$
  - Information goes with group velocity, “Einstein” is ok!
    \[ (\text{group velocity}) \times (\text{phase velocity}) = c^2 \] (in free space)
Resonant cavities

- **Fig 11 (top frame)**, example of the pill box RF cavity (side view, left, and end view, right)
- Beam goes along $z$-axis, cavity is $L$ units long.
- Symmetry suggests cylindrical coordinates $(r, \phi, z)$ shown
- Need longitudinal electric field $E$ (as shown) to accelerate particles
Resonant cavities

- An RF generator feeds waves into the cavity
- In this example, excites in the transverse magnetic mode (TM)
- Symmetry: Fields independent of $\phi$
- Boundary condition: $E(t) = 0$ at conductor at boundary, at $r = R_c$
- Bessel’s Equation is involved, the solution is

$$E_z = E_o J_0 \left( \frac{\omega}{c} r \right) \cos(\omega t + \xi) \quad (45)$$
Resonant cavities

The solution is

\[ E_z = E_o J_0 \left( \frac{\omega}{c} r \right) \cos \left( \omega t + \xi \right) \] \hspace{1cm} (45)

\( J_0 \) is the zero-order Bessel function! REALLY!

Must have \( J_0(\omega R_c/c) = 0 \) (boundary condition)

Occurs only at certain values of \( \omega R_c/c \), smallest (lowest frequency) one is:

\[ \omega_c = \frac{2.405c}{R_c} \quad \text{and} \quad f_c = \frac{2.405c}{2\pi R_c} \] \hspace{1cm} (46)
Resonant cavities

A good approximation to the Bessel function is:

\[ J_0 \left( \frac{\omega r}{c} \right) \approx \cos \left( \frac{\omega r}{2.405c} \pi \right) = \cos \left( 0.6531 \frac{\omega r}{c} \right) \]  

(47)

The magnetic field is axial, can use Eq. (10) to get \( B(t) \) from \( E(t) \)

Applying the right-hand rule,

if RF phase \( \xi \) is right for accelerating, magnetic field will help focus.

\( \omega_c \) is independent of \( L \! \)!

Example: \( f_c = 380 \text{ MHz for } R_c = 0.3 \text{ m} \)
Resonant cavities

Length $L$ is not irrelevant!
Determines the time particle gets accelerated
Leads to the transit time factor $T_f$

$$T_f = \frac{\text{energy added at peak field}}{\text{energy added if field were static}}$$

$$T_f = \frac{E_o \int_0^{L/2} dz \cos \left( \frac{\omega z}{\beta c} \right)}{E_o (L/2)} = \sin \left( \frac{\omega L}{2 \beta c} \right)$$ (48)

Example: At $\beta=1$, $T_f=0.9 \implies L/R_c \approx 2/3$

Panofsky equation gives the energy gain for gap crossing of particle with charge $q$: $\Delta W = qE_o \cos \xi L$ (49)
Resonant cavities

- Purity of any oscillatory situation, including RF and even radio, is characterized by the quality factor \( Q \)

\[
Q = \frac{\text{stored energy}}{\text{energy lost per radian of oscillation}} = \frac{\text{design frequency}}{\text{frequency separation between half-power points}} = \frac{\omega_0}{\Delta \omega}
\]

- Usually want highest possible value of \( Q \)
  - Energy efficiency
  - Tuning/performance goals

- Confusion of the terminology: RF quality factor, versus radiation protection quality factor
Resonant cavities

For pillbox, for surface resistivity (bulk resistivity divided by the skin depth) $\rho_s$,

$$Q = \frac{2.405 \mu_o c}{2 \rho_s [1 + (R_c / L)]}$$

(50)

For copper, $\rho_s = 10^{-8} - 10^{-9} \, \Omega$, for $f_c$ at about 400 MHz, $Q \approx 10^4$

Resistivity of niobium-based superconductors is about $10^5$ times smaller – Advantage: superconductors!

Many other modes than this pillbox are possible and are used!

The RF gradient (MV m$^{-1}$) is important; not the same as the quadrupole gradient

- Present limits are about 100 MV m$^{-1}$
- Typical room temperature copper cavities operated at about 20 MV m$^{-1}$
- Challenging to increase gradients
  - Example: SLAC uses 40 MW of line power to run two-mile linac at about 20 MV m$^{-1}$
  - Would take 25 times more power to run at 100 MV m$^{-1}$, if feasible from other considerations.
  - Or one nuclear power station!
Phase stability

See Fig. 11 (middle frame) – electric field as a function of arrival time of particles in an RF cavity

- Late particle gets a higher electric field, hence more kick
- Early particle gets a lower electric field, hence less kick
- Moves early and late particles closer to stable position at next RF gap
- Very, very desirable phenomenon!
Phase stability

If this did not work, would lose early and late particles!

Note ocean surfers don’t ride the tops of the waves either!

- Some observations
  - Doesn’t work if phasing puts stable particle at the maximum
    - late particles get less kick, get lost
  - Doesn’t work if stable particle arrives after maximum
    - lose both late and early particles
  - Doesn’t work on negative cycles unless deceleration wanted (This has been done for special purposes.)
- Each RF cycle creates a region in time consistent with stable motion, called RF buckets or just buckets.
- Particles occupying buckets are called bunches.
Fig. 11 (bottom frame) shows an Alvarez drift tube linac (DTL)
Lower energy beam (from lower energy accelerator enters from left
Entire structure (tank) is conductive, fed with RF
Linear accelerators – drift tube linacs

- Beam is “Faraday shielded” from RF except in the gaps.
- Acceleration occurs only in the gaps, not in the drift tubes.
- RF fields in all gaps are synchronized (in phase).
- \( D \) the lengths of the drift tubes in this full wave mode must satisfy
  \[
  D = \beta \lambda \quad (51)
  \]
- \( D \) is shorter at low velocities (nonrelativistic conditions, \( \beta \ll 1 \))
- \( \lambda \) is RF resonant wavelength,
- \( \beta c \) is the velocity at a given drift tube
- \( \lambda/c \) is the time duration of 1 RF cycle
- Works best for \( 0.04 < \beta < 0.4 \).

\[
\lambda = \frac{c}{f} = \frac{2\pi c}{\omega} \quad (44)
\]
Linear accelerators – drift tube linacs

- See why phase stability is needed?
- Quadrupole magnets can go in drift tubes to focus
- High beam currents (mA) available
- Wideröe design was first, half-wave tubes used
  - $Q$ of such design is lower than for Alvarez type
  - But, Wideröe design is effective for low velocity ions (e.g., $\beta < 0.03$) and low frequencies (e.g., $f < 100$ MHz)
Linear accelerators – drift tube linacs

Drawbacks

- No acceleration in DTs, wasted space
- Not good for electrons
- Focusing quads cannot fit in first few (short) DTs
- Resonant structure, cannot vary energy easily
- Need to watch for RF heating
  - Heating will distort structure, mess up RF resonance
  - Need water cooling
- Low duty factors (often only a few %)
- Get pulsed beam (always!)
  - May require a pulsed injector, to avoid “lossy” early stages
    - To avoid radioactivation
    - To avoid undesired beam heating
Linear accelerators – drift tube linacs

Fermilab 750 kev to 116 MeV DTL

Photos from Fermilab website
**Linear accelerators – coupled cavity linacs**

- As \( \beta \to 1 \), DTLs become inefficient and too long
- Velocity \( \beta \) is no longer changing
- **Coupled-cavity structures are better**
  - Side-coupled version is popular
  - Cavities excited along their sides
  - Other possibilities exist and are used
- **Standing wave (i.e., stationary mode) cavities**
  - Effective in domain \( 0.4 < \beta < 1 \)
  - Provide 2 gaps per \( \beta \lambda \)
  - Used for both ions and electrons
  - Can used for colliders since standing wave is a superposition of 2 traveling waves from opposite directions
- **Mechanical tolerances are important at large \( f \), small \( \lambda \).**
Linear accelerators – coupled cavity linacs

Fermilab 116 - 400 MeV Side-Coupled Linac

Photo by J. D. Cossairt, Fermilab exhibits, cut-away view

Photo from Fermilab website
**Linear accelerators – electron linacs**

- For electrons, at $T = 3 \text{ MeV}$, $\beta = 0.989$!
- Continuous, uniform wave guide looks tempting!
  - Phase velocity exceeds $c$, form of wave would race ahead of particles
  - Need longitudinal field component to accelerate, only have transverse fields if energy flow is longitudinal.

Recall:

$$S(t) = \frac{1}{\mu_o} E(t) \times B(t) \tag{43}$$

- **Solution**: Break up wave guide into periodic structures
  - Act like individual resonant cavities
  - Eliminates wavelengths that do not “fit”
  - Commonly accomplished with loaded circular disks
  - Resistance reduces phase velocity to below $c$
  - Called disk-loaded or iris-loaded waveguide
  - Can use high frequency RF, e.g., 3 GHz at SLAC
Linear accelerators – electron linacs

SLAC 2-Mile disk-loaded linac

W. Panofsky, photos from SLAC website
Linear accelerators – recirculating electron linacs

Recirculating linacs

• Line up sets of linacs to achieve desired energies in steps

• Intervening circular arcs of bending magnets direct the beam between stages

• Use linacs more than once ($\beta = 1$)

• Especially with superconducting RF, continuous wave mode (CW, near 100 % duty factor) becomes possible

The Continuous Electron Beam Accelerator Facility (CEBAF) is the most prominent example, operates in near continuous wave mode
Linear accelerators – CEBAF pictures

linac section

bending magnet arcs

Photos from TJNAF website
Radio-frequency Quadrupoles (RFQs)

- Radio-frequency quadrupoles (RFQs) commonly operate in the velocity range of $0.01 < \beta < 0.06$
- Combine the focusing capabilities of electric quadrupoles with the acceleration by RF fields.
- Are a type of linac.
- Used only for protons and ions, not electrons
- See Fig. 12
Radio-frequency Quadrupoles (RFQs)

- 4 vanes are excited by RF as with $E(t) = E_o \cos(\omega t + \xi)$, with $\omega = 2\pi f$ (42)
- Note modulation, characterized by modulation parameter $m$
- $yz$-plane modulation is out of phase with $xz$-plane modulation
- Unit cell length $L = \beta \lambda/2$ with $\lambda$ = RF wavelength
**Radio-frequency Quadrupoles (RFQs)**

- Reasons for modulation
  - Need longitudinal electric field to accelerate
  - Need electric field on the optic ($z$) axis to accelerate most of the beam
- Phase stability, etc. applies in an RFQ
- Get focusing and acceleration simultaneously
Radio-frequency Quadrupoles (RFQs)

- Some useful results

\[ \Delta W = \frac{q\pi AV_o I_o (kr) \cos \xi}{4}, \quad E_o = \frac{1}{L} \int_0^L E_z dz = \frac{2AV_o}{\beta\lambda}, \quad \text{and} \quad T_f = \frac{\int_0^L E_z \sin kzdz}{\int_0^L E_z dz} = \frac{\pi}{4}. \quad (52) \]

- \( \Delta W \) = energy gain per cell
- \( E_o \) = spatial average of peak axial accelerating field over the cell (average axial field)
- \( T_f \) = transit time factor [See Eq. (48) ]
- \( k = 1/\lambda \), the wave number
- \( A \) = accelerating efficiency [dependent on \( m \), \( A=0 \) for \( m=1 \)]
- \( V_o \) = the applied voltage
- \( AV_o \) = effective axial voltage over the length of the unit cell
- integrals are over the unit cell length
Radio-frequency Quadrupoles (RFQs)

\[ \Delta W = \frac{q \pi A V_o I_o (kr) \cos \xi}{4}, \quad E_o = \frac{1}{L} \int_0^L E_z dz = \frac{2 A V_o}{\beta \lambda}, \quad \text{and} \quad T_f = \frac{\int_0^L E_z \sin kzdz}{\int_0^L E_z dz} = \frac{\pi}{4}. \quad (52) \]

\( I_o (kr) \) is the modified Bessel function of the first kind of zero order. Really!

Well approximated by:

\[ I_o (kr) \approx 1 + \frac{(kr)^2}{4}. \quad (53) \]

Putting this all together

\[ \Delta W = q E_o T f I_o (kr) L \cos \xi \quad (54) \]

Note: more energy gain for off-axis particles \((r>0)\)

But, paths lengths for them are longer, phase stability works!
Radio-frequency Quadrupoles (RFQs)

• Some points
  • Ideal vanes are hyperbolae, but circular rods have been used (hyperbolae are difficult to make!)
  • Even trapezoidal longitudinal profiles, easier to make on a lathe have been used
  • Superconducting RFQs are successful
  • Have been used to store, not accelerate particles

• Advantages/Disadvantages
  • At low velocities, available electric field focusing is stronger than magnetic field focusing [remember the velocity factor in Eq. (11)]
  • RFQs have more compact infrastructure than some alternatives
  • Energy cannot be easily changed, a disadvantage
Radio-frequency Quadrupoles (RFQs)

RFQ on exhibit at Fermilab, photo by J. D. Cossairt
Radiation protection considerations (linacs)

• RF devices of considerable energy
• Electrical and non-ionizing radiation hazards
• Dark currents with associated production of x-rays Major RF components often external to beam enclosures, hazards are accessible to personnel
• Special peculiarities
  – multipacting with avalanches (back-and-forth acceleration across gaps)
  – electron field emission
  – undesired heating (adds electrical impedance, reduces $Q$)
• Single pass machines-What is the worst case accident?
• Effects of RF fields in radiation safety instrument response. Faraday shields needed?
Cyclotrons, synchrocyclotrons, & betatrons - cyclotrons

- Have the advantage of using the RF structure repeatedly
- Work best at low energies where
  - Relativistic considerations do not dominate
  - Synchrotron radiation is negligible
- See top frame of Fig. 13 for the classic cyclotron

E. Lawrence & M. Livingston
Photos from Nobel Prizes website
Cyclotrons

Start with Eq. (18) \[ \frac{mv^2}{R} = qvB \] (18)

We get \[ v = \frac{qB_o}{\gamma m_0 r} \] (55)

Taking the path as circular

the orbit period is \(2\pi r/v\) & the orbit frequency is \(v/2\pi r\)

Get synchrotron angular frequency \(\omega_{\text{syn}}\)

\[ \omega_{\text{syn}} = 2\pi \frac{qB_o}{2\pi \gamma m_o} = \frac{qB_o}{\gamma m_o} \] (56)

At low velocities have the similar cyclotron frequency \(\omega_{\text{cyc}}\)

\[ \omega_{\text{cyc}} = \frac{qB_o}{m_o}, \text{ for } \gamma \approx 1 \] (57)
**Cyclotrons**

Lack of radial dependence allows orbit to be isochronous for constant magnetic field $B_o$.

RF frequency must be an integer multiple, the harmonic number $h$ of the cyclotron frequency

$$\omega_{RF} = h\omega_{cyc} \quad (58)$$

Design constrains $h$

E.g., in classic cyclotron, $h$ has to be an odd number
**Cyclotrons**

- As particle energy increases, $\gamma$ increases
- Cannot just reduce RF frequency as implied by
- Will lose lower energy particles
- Using Eqs, (5), (18), & (56) [SI units]

$$\omega_{syn} = \frac{qB_o}{\gamma m_o} \quad (56)$$

$$\frac{mv^2}{r} = qvB \rightarrow \frac{p}{r} = qB \quad (18)$$

$$r = \frac{p}{\omega_{syn} \gamma m_o} = \gamma \frac{W_o}{\omega_{syn} \gamma m_o c} \left[ 1 - \left( \frac{W_o}{W(r)} \right)^2 \right]^{1/2} = \frac{\gamma m_o c^2}{\omega_{syn} \gamma m_o c} \left[ 1 - \left( \frac{W_o}{W(r)} \right)^2 \right]^{1/2}$$

$$= \frac{c}{\omega_{syn}} \left[ 1 - \left( \frac{W_o}{W(r)} \right)^2 \right]^{1/2} \quad (59)$$
Cyclotrons

\[
r = \frac{p}{\omega_{\text{syn}} \gamma m_o} = \frac{\gamma}{\omega_{\text{syn}} \gamma m_o} \frac{W_o}{c} \left[1 - \left(\frac{W_o}{W(r)}\right)^2\right]^{1/2} = \frac{\gamma m_o c^2}{\omega_{\text{syn}} \gamma m_o} \left[1 - \left(\frac{W_o}{W(r)}\right)^2\right]^{1/2}
\]

\[
= \frac{c}{\omega_{\text{syn}}} \left[1 - \left(\frac{W_o}{W(r)}\right)^2\right]^{1/2}
\]

(59)

- To achieve synchronization, want to scale magnetic field strength with \( r \)
- Solve Eq. (59) for total energy \( W(r) \),

\[
W(r) = W_o \left[1 - \left(\frac{\omega_{\text{syn}} r^2}{c^2}\right)\right]^{-1/2}
\]

, (60) then

\[
B_z = \frac{\omega_{\text{syn}} \gamma m_o}{q} = \frac{\omega_{\text{syn}} W(r)}{q c^2} = \frac{\omega_{\text{syn}} W_o}{q c^2} \left[1 - \left(\frac{\omega_{\text{syn}} r^2}{c^2}\right)\right]^{-1/2} = \frac{\omega_{\text{syn}}}{q} m_o \left[1 - \left(\frac{\omega_{\text{syn}} r^2}{c^2}\right)\right]^{-1/2}
\]

(61)
Cyclotrons

\[ B_z = \frac{\omega_{\text{syn}} \gamma m_o}{q} = \frac{\omega_{\text{syn}} W(r)}{qc^2} = \frac{\omega_{\text{syn}}}{qc^2} W_o \left[ 1 - \left( \frac{\omega_{\text{syn}}^2 r^2}{c^2} \right) \right]^{-1/2} = \frac{\omega_{\text{syn}}}{q} m_o \left[ 1 - \left( \frac{\omega_{\text{syn}}^2 r^2}{c^2} \right) \right]^{-1/2} \]  

(61)

Useful to use particle or ion masses in atomic mass units \( m_u \) (1.0

\[ m_u = 931.494 \text{ MeV} \text{ c}^{-2} = 1.66054 \times 10^{-27} \text{ kg} \]

& beam particle charge \( q \) in units of electronic charge by using Eq. (19)

\[ R(\text{meters}) = \frac{p}{qB} = \frac{p(\text{MeV c}^{-1})}{299.79qB} \]  

(19)

\[ B_z = \frac{\omega_{\text{syn}} 931.49}{q 299.79} A \left[ 1 - \left( \frac{\omega_{\text{syn}}^2 r^2}{c^2} \right) \right]^{-1/2} = 3.107 \frac{\omega_{\text{syn}}}{q} A \left[ 1 - \left( \frac{\omega_{\text{syn}}^2 r^2}{c^2} \right) \right]^{-1/2} \]  

(62)

where \( A \) is the atomic mass number, \( B_z \) remains in Tesla

Traditionally achieved using trim coils to increase \( B_z \) with \( r \)
**Cyclotrons**

- **Result:** isochronous cyclotron
- **Apollo 13:** “Houston, we have a problem!”:
  - Field index $n<0 \Rightarrow$ radial component $B_r$ increases with $r$
  - Makes orbits unstable vertically (in the $z$-component)
- **Solution:** Create hills and valleys in the orbit plane, use sector and edge focusing) (see Fig. 13 lower frame)
- **Alternative name:** azimuthal varying field (AVF) cyclotron

![Diagram of sector focusing and spiral-ridge focusing](image)
Cyclotrons

Sector magnets in cyclotrons: Indiana University Cyclotron Facility (IUCF)

Injector

Main Stage

Photos by J. D. Cossairt

R. Pollock, photo from IUCF website
Cyclotrons

Example of conventional spiral ridge cyclotron:
88” Cyclotron at LBNL

Photo from LBNL website
Cyclotrons

A superconducting spiral ridge cyclotron K-1200 Cyclotron at National Superconducting Cyclotron Laboratory Michigan State University

Photos of K1200 and H. Blosser from NSCL website, labeled photo of interior of K1200 from R. Ronningen, NSCL
Cyclotrons

- Flutter amplitude $F_l$ measures differences between peak field in hills $B_h$ & lowest field in valleys $B_v$
  \[ F_l = \frac{B_h - B_v}{2B_{ave}} \]  

- Momentum of extracted particles at radius $R$ is $p=\gamma m_o \beta c$

- Kinetic energy is
  \[ T(R) = m_o c^2 (\gamma - 1) = \frac{p^2(R)}{(\gamma + 1)m_o} \]  

- Parameter $K$ for ions of atomic mass number $A$ and charge $q$ is useful
  \[ K = \frac{T(R)}{A} = \frac{q^2 \gamma^2 B_o^2 R^2}{(\gamma + 1)m_o} \]  

  (SI units) $= \frac{96.48 \gamma^2 (B_o R)^2}{(\gamma + 1) \frac{q^2}{A}} \left( \frac{\text{MeV}}{\text{amu}} \right)$

- If RF voltage drop is $V_o$ (MV), after $n$ turns, $T_n = qN V_o$ (MeV)

Solving Eq. (18) for orbit radius $r_n$

\[ r_n = \frac{p_n}{qB} = \frac{[(\gamma + 1)m_o AqNV_o]^{1/2}}{qB} = \frac{\sqrt{931.49}[(\gamma + 1)AqNV_o]^{1/2}}{299.79qB} = \frac{0.1018}{B} \left( \frac{(\gamma + 1)AV_o}{q} \right)^{1/2} N^{1/2} \]
Synchrocyclotrons

• Instead of increasing field with radius, modulate the RF frequency with $\gamma$ according to Eq. (56):

\[
\omega_{\text{syn}} = 2\pi \frac{qB_o}{2\pi \gamma m_o} = \frac{qB_o}{\gamma m_o} \tag{56}
\]

• Protons of 500 MeV to 1 GeV kinetic energy typical
• Poor duty factor (only one group of protons at a time)
• Cycle times often line power frequencies or submultiples (in USA, 60 Hz, 30 Hz, 15, Hz, etc.)
• Largely obsolete, superseded by
  – Modern isochronous cyclotrons (lower energies)
  – synchrotrons (higher energies)
Synchrocyclotrons

- Largest remaining; 1 GeV Petersburg Nuclear Physics Institute, Russia (mass = 1 Eiffel Tower, 7-10 kTonne)
Betatrons

- Due to strong dependence on $\gamma$, cyclotrons are inappropriate for electrons.
- Historic solution was the betatron
- Now nearly completely superseded by linacs and synchrotrons
- Despite obsolescence will discuss here because the betatron illustrates important features inherent in all circular accelerators
- Only accelerator type that use electromagnetic induction to operate.
- Electrons are injected from some lower energy stage into an orbit of radius $R$.
- Acceleration occurs in this orbit of constant radius.
- Have a ramped field, often at “line power” frequencies (USA = 60 Hz or submultiples)
- Duty factor is similar to that of synchrocyclotrons
Betatrons

Start with Eq. (9)

\[ \oint E \cdot dl = -\int_s \frac{dB}{dt} \cdot dS \quad (67) \]

Relates integral of electric field \( E \) around orbit to rate of change of integral of rate of change of magnetic field inside orbit \( dB/dt \)

Symmetry: \( E \) is not dependent on \( \theta \)

Eq. (67) becomes:

\[ 2\pi RE(R) = \pi R^2 \frac{dB_{ave}}{dt} \quad (68) \]
Betatrons

\[ 2\pi RE(R) = \pi R^2 \frac{dB_{ave}}{dt} \quad (68) \]

Solving for the electric field,

\[ E(R) = \frac{1}{2} \frac{dB_{ave}}{dt} R \quad (69) \]

Remember Eq. (11): \[ F = q(v \times B + E) = \frac{dp}{dt}, \]

Leads to the equation of motion:

\[ \frac{dp}{dt} = \frac{1}{2} qR \frac{dB_{ave}}{dt} \quad (70) \]

Betatron condition: guide field \( B_g \) at orbit is \( \frac{1}{2} \) average field enclosed by the orbit. Hence, for unit charge \( (q=1) \), and \( B_g R \) (T m):

\[ p = qB_g R \text{ (SI units)} = 299.79 B_g R \text{ (MeV c}^{-1}) \quad (71) \]
Betatrons

Need to consider transverse stability
Field index \( n > 0 \) makes focusing plausible

Symmetry: \( \mathbf{B} \) has two components; \( B_z \) and \( B_r \) \((B_\theta = 0)\)

Eq. (11): \( F = q(v \times B + E) = \frac{dp}{dt} \)
gives an equation of motion in cylindrical coordinates \((r, \theta, z)\), knowing that \( p = mv \)

\[
\frac{d^2 r}{dt^2} = \left[ \frac{d^2 r}{dt^2} - r \left( \frac{d \theta}{dt} \right)^2 \right] \hat{r} + \frac{d^2 z}{dt^2} \hat{z} = -q \frac{vB_z}{m} \hat{r} - q \frac{vB_r}{m} \hat{z} \tag{72}
\]

For small transverse deviations from the orbit:
\( \omega = \frac{d \theta}{dt} = v/R = qB/m \)
Betatrons

Using the field index, \[ B_r \approx \frac{-nB_o z}{R} \] (35)

From Eq. (72), equate the axial \((z)\) components

\[ \frac{d^2 z}{dt^2} = -\frac{qB_g}{m} \frac{v}{R} nz = -\omega^2 nz \] (73)

Get harmonic oscillator solution:

\[ z(t) = z_{max} \cos \left[ (\omega \sqrt{n}) t + \phi \right] \] (74)

Angular frequency is \(\omega n^{1/2}\)

Amplitude is \(z_{max}\)

Eq. (73) is that of an ordinary steel spring.
Implies stability, good news indeed!
**Betatrons**

Radial \((r)\) component is more difficult

Consider small deviations \(\zeta\) from orbit \(r = R + \zeta\)

Equating components for Eq. (72):

\[
\left[ \frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right] = \left[ \frac{d^2 r}{dt^2} - \frac{v^2}{r} \right] = -q \frac{v}{m} B_z(r) \tag{75}
\]

Substituting:

\[
\frac{d^2 \zeta}{dt^2} - \frac{v^2}{R + \zeta} = -q v B_z(R + \zeta) \tag{76}
\]

Approximating:

\[
\frac{1}{R + \zeta} = \frac{1}{R} \left( 1 + \frac{\zeta}{R} \right)^{-1} \approx \frac{1}{R} \left( 1 - \frac{\zeta}{R} \right) \tag{77}
\]

\[
\frac{d^2 \zeta}{dt^2} - \frac{v^2}{R} \left( 1 - \frac{\zeta}{R} \right) = -q \frac{v}{m} B_z(\zeta) \tag{78}
\]

Using \(\frac{mv^2}{R} = qvB\) (18) \(\Rightarrow \frac{v^2}{R} = \frac{qvB}{m} \),

\[
\frac{d^2 \zeta}{dt^2} + \left[ B_z(\zeta) - B_g \right] \frac{qv}{m} + \frac{qvB_g}{m} \frac{\zeta}{R} = 0 \tag{79}
\]
**Betatrons**

Do Taylor expansion, $B_z = B_g + \zeta dB/dr + \ldots$

Hence $B_z - B_g$ can be approximated by $\zeta dB/dr$

Make other substitutions;

$v = \omega R$

$q/m = \omega / B_g$

$n = -R dB/(B_g dr)$

$$\frac{d^2 \zeta}{dt^2} + \frac{R}{B_g} \frac{dB}{dr} \omega^2 \zeta + \omega^2 \zeta = 0 \rightarrow \frac{d^2 \zeta}{dt^2} + \omega^2 (1-n) \zeta = 0 \quad (80)$$

Solution is also a harmonic oscillator:

Angular frequency is $\omega(1-n)^{1/2}$

Amplitude is $\zeta_{max}$

We have stability in both planes!

$$\zeta(t) = \zeta_{max} \cos\left[(\omega\sqrt{1-n})t + \phi\right] \quad (81)$$
Betatrons

• These are axial and radial betatron oscillations.
• Eqns. (73) and (80) are the Kerst-Serber Equations.
• Get a fixed number of betatron oscillations per orbit (not usually integers), called the tune
  – axially, \( n_{z} = n^{1/2} \)
  – radially, \( n_{r} = (1-n)^{1/2} \)
• Orbits are unstable if \( 0 < n < 1 \) is not satisfied
• \( n=0 \) or \( n=1 \) implies an unstable drift in either \( z \) or \( r \)
• \( n>1 \) implies absorption (i.e., beam loss)
• \( n<0 \) implies radial defocusing
• \( n=1/2 \) has tunes same in both coordinates, explains the unique focal properties of the \( n=1/2 \) magnet
• Have similar oscillations in other circular accelerators
Radiation protection considerations of cyclotrons, synchrocyclotrons, and betatrons

- RF sources can produce x-rays
- Deflection devices; inflectors and electrostatic septa, can produce x-rays.
- Injector beam emittance needs matching to the acceptance of higher energy stage, otherwise have beam loss, activation, etc.
- Orbits need careful tuning upon injection and extraction to minimize radioactivation of components.
- Synchrotron radiation with consequent x-ray production can be a consideration at betatrons
- Massive magnets can present radioactive material handling problems at the time of decommissioning.
- Large copper components can be the locus of long-lived radioactivation (e.g., $^{60}\text{Co}$).
Synchrotrons

- Ultimate large circular accelerator at present
- Strong-focusing version is dominant
- Limitations of other circular machines lead to this result
  - Magnetic field inside orbit of betatron serves no other purpose, though needed for magnetic induction,
  - Synchrocyclotron magnets become impractically large at high energy
  - Need small beam, => strong focusing
- Operating Concept
  - Accelerate in an orbit of constant $R$ (like in a betatron!)
  - Inject particles into the ring from a lower energy machine
    - Sufficient injection energy required to assure “good” magnetic field free of eddy currents, hysteresis effects, power supply ripple at low current, etc.
  - Use RF, rather than induction, to accelerate beam
  - Synchronously increase magnetic field in orbit plane $B$ to match $p$
  - Explains the name: synchrotron
Synchrotrons-weak focusing

- Cyclotrons, synchrocyclotrons, & betatrons all are weak-focusing
- Apertures scale with $R$ to achieve necessary vertical focusing
- Example: 6 GeV Bevatron at Lawrence Radiation Laboratory (now LBNL)
  
  \[ n = 0.6 \Rightarrow \text{aperture: 1.16 m (horizontal) by 0.30 m (vertical)}! \]
Synchrotrons-strong focusing

- Limitations recognized early, particularly by R. R. Wilson (R. R. Wilson, photo from Fermilab website, D. Edwards, photo from Cornell U. website)
- Applied immediately to Cornell 1 GeV electron synchrotron (early 1950’s) (Wilson redesigned ring to strong-focusing after magnets were built, Edwards was the student who did the work!)
- How to do? Alternate magnets like in Fig. 5 to get focusing effect
Synchrotrons—strong focusing, combined function

- BNL 30 Alternating Gradient Synchrotron (AGS) largest example
- For AGS, $n = 357$
- Limitation: Making these magnets, too, becomes expensive

3 GeV Cosmotron vs AGS Magnets       View of AGS

Drawing and photo from BNL website
Synchrotrons-strong focusing

- Separated function strong-focusing is current state-of-the-art
- Called FODO, focusing-defocusing lattice
- See Fig. 15 (top frame)
- Each “magnet” can be a string of near-identical magnets
- Focusing separated from bending (not true in AG)
- Longitudinally, magnets are rectangular in cross section, for large $R$. 

**FODO SYNCROTRON - PLAN VIEW**

- RF
- focusing quad (horizontal)
- dipole
- defocusing quad (horizontal)
- R
- Straight Section
Synchrotrons—strong focusing

- **Straight Sections**
  - Any number
  - Machine functions
    - Injection
    - Extraction
    - RF
    - Diagnostics
    - Insertion devices (at light sources)
  - Experiments

Remainder of discussion will be vignettes, see references for details!
Synchrotrons-examples

Fermilab 8 GeV Booster

Photos from Fermilab website
Synchrotrons-examples

Fermilab 150 GeV Main Injector  Fermilab 1 TeV Superconducting Tevatron

(R. Lundy, R. Orr, H. Edwards, & A. Tollestrup)
Photos from Fermilab website
**Synchrotrons-phase stability and transition crossing**

Ideal particle on reference orbit will pass between 2 markers (e.g., complete orbit) in time $\tau = L/v$ ($L =$ path length, $v =$ velocity)

If ideal path is NOT followed we have a deviation in path length $\Delta L$ and/or a deviation in speed $\Delta v$

Get deviation in time interval $\tau$

$$\frac{\Delta \tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta v}{v} \quad (82)$$

Intuitive! $\tau$ increase with path length, decreases with increased speed

$\Delta L/L$ is proportional to relative deviation in momentum $\Delta p/p$ through a dispersion coefficient $\gamma_t$:

$$\frac{\Delta L}{L} = \frac{1}{\gamma_t^2} \frac{\Delta p}{p} \quad (83)$$

Higher momentum usually implies a longer path

$\gamma_t$ is also called the transition gamma
Synchrotrons-phase stability and transition crossing

Want to connect $\Delta p/p$ to $\Delta \beta/\beta = \Delta \nu/\nu$

Take differential of $p$ and rewrite Eq. (82):

$$\frac{\Delta p}{p} = \frac{\Delta \left[ (1 - \beta^2)^{-1/2} \beta m_o c \right]}{\gamma \beta m_o c} = \frac{\left( (1 - \beta^2)^{-1/2} + \beta^2 (1 - \beta^2)^{-3/2} \right) \Delta \beta}{\gamma \beta}$$

$$= \gamma \left[ 1 + \frac{\beta^2}{1 - \beta^2} \right] \Delta \beta = \gamma^2 \frac{\Delta \beta}{\beta} \quad (84)$$

$$\frac{\Delta \tau}{\tau} = \left[ \frac{1}{\gamma^2_t} - \frac{1}{\gamma^2} \right] \frac{\Delta p}{p} = \eta \frac{\Delta p}{p} \quad (85)$$

$\eta$ is the phase slip factor
Synchrotrons-phase stability and transition crossing

\[ \frac{\Delta \tau}{\tau} = \left[ \frac{1}{\gamma^2} - \frac{1}{\gamma_t^2} \right] \frac{\Delta p}{p} = \eta \frac{\Delta p}{p} \]  \hspace{1cm} (85)

The transition energy \( T_t \) corresponds to \( \gamma = \gamma_t \)

At \( T = T_t \), \( \eta = 0 \) and phase stability changes.

See Fig. 15, middle frame
Synchrotrons-phase stability and transition crossing

- Below transition, a particle with $\Delta p/p > 0$ has greater speed, gets around quicker the next time.
- Above transition, a particle with $\Delta p/p > 0$ takes a longer path, takes longer to get around.
- Phenomena is due to special relativity; $\beta = 1$.
- At transition:
  - $\tau$ is independent of momentum.
  - All particles are isochronous (same lap times!)
  - There is no phase stability. (bad news-beam losses!)

\[ \eta < 0 \]

\[ \eta > 0 \]
Synchrotrons-phase stability and transition crossing

- Accomplishing transition crossing cleanly is important at most proton synchrotrons
  (All electron synchrotrons operate above transition, no problem)
- Done by inducing change in RF phase at transition, usually quickly

- Examples at Fermilab
  - Fermilab Booster \((T=0.4-8 \text{ GeV}, R=74.47 \text{ m}) \gamma_t = 5.45\) \((T=4.17 \text{ GeV})\)
  - Fermilab Main Injector \((T=8-150 \text{ GeV}, R=528.3 \text{ m}) \gamma_t = 20.4\) \((T=18.2 \text{ GeV})\).
  - Superconducting Tevatron \((T=150-980 \text{ GeV}, R=1000 \text{ m}) \gamma_t = 18.7\) \((T=16.6 \text{ GeV})\). The Tevatron always operates above transition.
Synchrotrons-phase stability and transition crossing

• In linacs, $\eta < 0$ since path length is same for all particles so transition crossing is not an issue
• From this perspective, an isochronous cyclotron operates at transition.
Synchrotrons-longitudinal emittance

• The confinement of particle beams in RF bunches in synchrotrons needs quantification.

• Consider a 2-dimensional phase space
  
  – Variables are time $t$ and kinetic energy $T$
  
  – Phase space area is defined by $\Delta t \Delta T$

• Acceptance $A$ is the bucket “area” available to “hold” the beam

• If $T_s$ is the kinetic energy of the ideal particle (MeV) and $V_s$ is the gap voltage (MV),

$$A = \frac{16 \beta}{\omega_{RF}} \sqrt{\frac{qVT_s}{2\pi\hbar|\eta|}}$$  (86)

• $V_s$ typically has values of a few MV
Synchrotrons-longitudinal emittance

• If \( \Delta \xi \) (radians) is the maximum of small oscillations in phase angle at injection or maximum energy (non-accelerating conditions), the longitudinal emittance of the beam is

\[
S = \frac{\pi \beta \left(\Delta \xi\right)^2}{\omega_{RF}} \sqrt{\frac{qVT_s}{2\pi h|\eta|}} \tag{87}
\]

• For accelerating conditions
  – Bucket area shrinks to zero as \( \xi \to \pi/2 \)
  – Must have \( A > S \) to keep the particles
  – Otherwise, the particles are lost
• Examples of nominal longitudinal emittances:
  – Fermilab Booster 0.25 eV s
  – Fermilab Main Injector: 0.2 eV s
  – Fermilab Tevatron (collider mode): 3 eV s
**Synchrotrons-harmonic number**

Same concept as for cyclotrons & synchrocyclotrons

\[ \omega_{\text{rf}} \tau_{\text{orbit}} = h2\pi \] is the number of RF cycles per orbit

- \( h \) is the harmonic number
- \( \tau_{\text{orbit}} \) is the orbit period
- \( h \) is an integer, generally large

The ideal particle will travel around the reference orbit (i.e., through the magnet centers) with frequency

\[ f_{\text{orbit}} = \frac{1}{\tau_{\text{orbit}}} = \frac{f_{\text{RF}}}{h} \quad (88) \]

Establishing the closed orbit is important at all synchrotrons
Requires correction to offset imperfections, mistunings, etc.

Examples of nominal harmonic numbers:
- Fermilab Booster: \( h = 84 \)
- Fermilab Main Injector: \( h = 588 \)
- Fermilab Tevatron (collider mode): \( h = 1113 \)
Synchrotrons-betatron oscillations and transverse emittance

- Transverse oscillations about reference orbits in synchrotrons are still called betatron oscillations
- More complex mathematically in synchrotrons, detailed derivations beyond the scope of this lecture
- Only will consider separated function synchrotrons, similar for all strong-focusing machines
- See Fig. 15 (top and bottom frames), define \((x, y, s)\) coordinates
Generalizing Kerst-Serber equations (from betatrons), and recognizing the change in naming of coordinate system:

\[
\frac{d^2 \xi}{dt^2} + \omega^2 (1-n) \xi = 0 \quad (80) \quad \xi \rightarrow x \quad x'' + K_x(s)x = 0 \quad (89)
\]

\[
\frac{d^2 z}{dt^2} + \omega^2 nz = 0 \quad (73) \quad z \rightarrow y, \quad y'' + K_y(s)y = 0 \quad (90)
\]

Eqns. (89) & (90)

- Use shorthand “prime” notation, \( x' = \frac{dx}{ds}, \ x'' = \frac{d^2x}{ds^2} \)
- Spring “constants” \( K(s) \) are position-dependent, not actual constants as in betatrons
- Not surprising! Focusing strong at quadrupoles, non-existent at dipoles
- In alternating-gradient synchrotron, values of \( K(s) \) switch signs at each focusing element
Synchrotrons-betatron oscillations and transverse emittance

\[ x'' + K_x(s)x = 0 \quad (89) \]
\[ y'' + K_y(s)y = 0 \quad (90) \]

Eqns. (89) & (90)
- Are examples of Hill’s Equation (studied in early 19th century for no practical purpose)
- The “forces” or “spring constants”, further, are given by

\[ K_x(s) = \frac{1}{BR} \frac{\partial B_y(s)}{\partial x} + \frac{1}{R^2} \quad (91) \]
\[ K_y(s) = -\frac{1}{BR} \frac{\partial B_y(s)}{\partial x} \quad (92) \]

- These look like quadrupole gradients, as they should.
- For completeness have included the centripetal \((1/R^2)\) term in Eq. (91), usually negligible at large accelerators.
- Often represented as constants within a given beam element or drift space between beam elements.
Synchrotrons-betatron oscillations and transverse emittance

\[ x'' + K_x(s)x = 0 \quad (89) \]
\[ y'' + K_y(s)y = 0 \quad (90) \]

\( K(s) \) values are often periodic in a lattice of repeated, identical elements:
\[ K(s + C) = K(s) \quad (93) \]

Often x- and y-dependencies are similar if not identical
Thus, one sometimes just writes down one coordinate to represent both

With arbitrary integration constants \( \varepsilon \) and \( \delta \), the solution for \( x(s) \) is
\[ x(s) = \sqrt{\varepsilon x} \beta_{CS,x}(s) \cos[\psi_x(s) + \delta] \]
and \[ \frac{d\psi_x(s)}{ds} = \frac{1}{\beta_{CS,x}(s)} \quad (94) \]

\( \varepsilon \) turns out to be the transverse emittance for this type of beam (i.e., in a synchrotron)
Synchrotrons-betatron oscillations and transverse emittance

\[ x(s) = \sqrt{\varepsilon_x \beta_{CS,x}(s)} \cos[\psi_x(s) + \delta] \] and \[ \frac{d\psi_x(s)}{ds} = \frac{1}{\beta_{CS,x}(s)} \] \hspace{1cm} (94)

- \( \beta_{CS,x}(s) \) and \( \beta_{CS,x}(s) \) values are
  - Called the amplitude functions or beta functions
  - One of a set of 3 Courant-Snyder parameters
    - (The CS subscript is nonstandard, used here to avoid confusion with \( \beta = v/c \))
    - Have units of length
- Examples of \( \beta_{CS,x}(s) \) values
  - Fermilab Booster: horizontal = 34 m, vertical = 21 m
  - Fermilab Main Injector: both = 58 m
  - Fermilab Tevatron (collider mode): both = 100 m

E. Courant, photo from BNL website
Synchrotrons-betatron oscillations and transverse emittance

• Some number of oscillations in $x$ or $y$ are found in an orbit
• As the betatron, this # is generally not an integer and is called the tune:

$$v_x = \frac{1}{2\pi} \oint ds \frac{d}{\beta_{CS,x}(s)}$$ (95)

• Tune values are managed to achieve desired properties
  – At injection
  – At transition
  – During colliding beams
  – At extraction

• Examples of tune values
  – Fermilab Booster: both about 6.7
  – Fermilab Tevatron (collider mode): both about 19.4
Synchrotrons-betatron oscillations and transverse emittance

- The slope, indicative of the mean angular spread in direction is important.
- Represented by \( x'(s) = dx(s)/ds \):
  \[
  x'(s) = -\frac{\varepsilon}{\sqrt{\beta_{CS,x}(s)}} \sin[\psi(s) + \delta] + \left[ \frac{\beta'_{CS,x}(s)}{2} \right] \sqrt{\frac{\varepsilon}{\beta_{CS,x}(s)}} \cos[\psi(s) + \delta] \quad (96)
  \]
- \( x(s) \) and \( x'(s) \) form the parameters of transverse phase space
- \( \Psi(s) \), reflective of the focusing strength, will be periodic
- \( x(s) \) and \( x'(s) \) cycle through sets of values at different locations
- Need remaining 2 Courant-Synder parameters (without \( x \) and \( y \) subscripts)

\[
\alpha_{CS}(s) = -\frac{\beta'_{CS}(s)}{2} \quad (97)
\]

\[
\gamma_{CS} = \frac{1 + \alpha_{CS}^2}{\beta_{CS}} \quad (98)
\]
Synchrotrons—betatron oscillations and transverse emittance

- Phase space $xx'$ in the synchrotron is bounded by the ellipse of Fig. 16 (upper frame)
  \[
  \frac{\dot{\varepsilon}}{\pi} = \gamma_{CS} x^2 + 2\alpha_{CS} xx' + \beta_{CS} x'^2 \quad (99)
  \]

- Commonly approximate particle distributions by Gaussian functions, e.g.
  \[
  n(x)dx = \frac{dx}{\sigma\sqrt{2\pi}} \exp\left[-x^2 / 2\sigma^2\right] \quad (100)
  \]

- These distributions are considered as stationary in time (i.e., assumes small orbit-to-orbit changes)
Synchrotrons-betatron oscillations and transverse emittance

• Connect the standard deviation $\sigma$ with the emittance $\epsilon$:

$$\epsilon = -\frac{2\pi\sigma^2}{\beta_{CS}} \ln[1 - F] \quad (101)$$

• $F$ is the fraction of particles contained with the ellipse of Eq. (99).

• Unfortunately conventions in this are not uniform!
  ✓ $\epsilon = \sigma^2/\beta$ (15%) – commonly used at electron accelerators
  ✓ $\epsilon = \pi \sigma^2/\beta$ (39%) – commonly used at proton accelerators
  ✓ $\epsilon = 4\pi \sigma^2/\beta$ (87%) – commonly used to characterize “complete” containment of beam
  ✓ $\epsilon = 6\pi \sigma^2/\beta$ (95%) – commonly used to characterize (really) “complete” containment of beam
Synchrotrons-betatron oscillations and transverse emittance

- See Fig. 16 (both frames)
- Shows evolution of beam ellipse containing particles during a segment of the lattice
  - Maximum displacement is
  - Maximum slope (angle) is

\[
x_{\text{max}} = \sqrt{\frac{\varepsilon \beta_{CS,\text{max}}}{\pi}} \quad (102)
\]

\[
x'_{\text{max}} = \sqrt{\frac{\varepsilon \gamma_{CS,\text{max}}}{\pi}} \quad (103)
\]
Synchrotrons-betatron oscillations and transverse emittance

• For uniform, circular half-aperture of radius $a$, the admittance is
  $$\frac{\pi a^2}{\beta_{CS,\text{max}}}$$
• During acceleration, emittance shrinks (due to relativity)
• But, normalized transverse emittance $\varepsilon_N$ is invariant $\varepsilon_N = \gamma \beta \varepsilon$ (104)
• Synchrotron radiation reduces transverse emittance
• Transverse emittance is a property of all particle beams
• Examples of $\varepsilon_N$ values
  – Fermilab Booster: 8 $\pi$ mm milliradians
  – Fermilab Main Injector: 30 $\pi$ mm milliradians
  – Fermilab Tevatron (collider mode): 24 $\pi$ mm milliradians
Synchrotrons - extraction

- Extraction of beam from synchrotrons is done as
  - single-turn extraction, use “kicker” magnets with fast time constants matched to the orbit period
  - resonant extraction, amplitude function is manipulated to get large beam sizes suitable for splitting by septa.
- Electrostatic septa often followed by magnetic elements get the beam into the desired extraction channel.
**Synchrotrons - extraction**

- At large accelerators, the magnetic elements may be Lambertson dipoles that follow electrostatic septa
  - Lambertson magnet has regular gap plus a field-free hole in its return yoke
  - Gap field deflects one beam
  - Field-free hole in a return yokes allows passage of the undeflected beam fraction
  - Get 2 separated beams

Photo from Fermilab Report No. FERMILAB-Conf-03/115 July 2003
G Lambertson, photo from LBNL website
**Colliding Beams**

- Special relativity limits energy available in particle collisions to study fundamental processes
- As energy increases, most goes into increased mass

\[ m = \frac{m_o}{\sqrt{1 - \beta^2}} = \gamma m_o \quad (3) \]

- Head-on collisions can use all the beam energy
- Hence, the prominence of colliding beam experiments at the energy frontier
Colliding Beams

- Luminosity is a crucial concept
  - In beam collisions with solid targets, interaction rates are proportional to the product of 3 quantities
    - Delivery rate of beam particles (i.e., particles s⁻¹)
    - Density of target nuclei \( \rho \frac{N_A}{A} \) (i.e., nuclei cm⁻³)
      - \( \rho \) is material density (g cm⁻³),
      - \( A \) is atomic mass number,
      - \( N_A \) is Avogadro’s Number (6.02 X 10²³ atoms g-mole⁻¹)
    - Reaction probability (the cross section, here in cm²)
  - Consider a 1 μA beam, singly-charged beam (6.25 X 10¹² s⁻¹) of 1 cm² cross-sectional area
  - Hits a solid target of about 1 cm thickness
  - Intensity X target nuclei density is of order 10^{35} cm⁻²s⁻¹.
  - Colliding two such 1 μA beams; only get about 4 x 10^{25} cm⁻²s⁻¹
  - Big Difference! Is this hopeless?
Colliding Beams

• Is this hopeless? No!
  – In circular machine get many orbits of same particles
  – With linacs or synchrotrons, can squeeze beam spot sizes
  – Will emphasize synchrotrons

• Working in cgs units as is usual, connect
  – Average rate of events $R_{\text{event}}$ (s$^{-1}$)
  – Luminosity $L$ (cm$^2$s$^{-1}$)
  – Reaction cross section of interest $\sigma_{\text{int}}$ (cm$^2$)

\[ R_{\text{event}} = L \sigma_{\text{int}} \] (105)

• If have 2 bunches of beam with $n_1$ and $n_2$ particles colliding at frequency $f$

\[ L = f \frac{n_1 n_2}{4\pi \sigma_x \sigma_y} \] (106)
Colliding Beams

- If have 2 bunches of beam with $n_1$ and $n_2$ particles colliding at frequency $f$
  \[ L = f \frac{n_1 n_2}{4\pi\sigma_x \sigma_y} \quad (106) \]
  - Where the two beam sizes are explicitly included
  - Lattice properties of the 2 beams are identical (i.e., in $x$ and $y$).

- Taking for this purpose $\varepsilon = \pi \sigma^2 / \beta$

- Denote the value of $\beta_{CS}(s)$ at the interaction point by $\beta^*$
  \[ L = f \frac{n_1 n_2}{4\sqrt{\varepsilon_x \beta^*_{CS,x} \varepsilon_y \beta^*_{CS,y}}} \quad (107) \]

- Get largest $L$ by maximizing beam particles and by minimizing $\varepsilon$ and $\beta^*$
Colliding Beams

- Get largest $L$ by maximizing beam & minimizing $\varepsilon$ and $\beta^*$

\[
L = f \frac{n_1n_2}{4\sqrt{\varepsilon_x\beta^*_{CS,x}\varepsilon_y\beta^*_{CS,y}}} \quad (107)
\]

- Minimize $\beta^*$ by
  - Using strong quadrupoles adjacent to interaction regions
  - Often not used during acceleration, only during collisions
  - Accept tradeoff of greater angular divergence since $\varepsilon$ conserved
  - Angular divergence can be focused back on after collisions

- Example: Before 1993, at Fermilab Tevatron $\beta^* = 0.5$ m was achieved (compared with $\beta_{CS,max}(s) = 100$ m in the general lattice. (Ongoing efforts continue to improve this!)

- Can see challenges of single-pass linear colliders!
**Colliding Beams**

- **Luminosity in a store decays with time**
  - Loss from collisions
  - Loss due to imperfections (beams are not perfect Gaussians)
  - Space charge effects
  - Beam-beam interactions (One beam looks like an electric current to the other!)

- **Integrated luminosity (integral over time) is important**
  - Cross sections can be expressed in units of picobarns (pb)
  - $1.0 \text{ pb} = 1 \times 10^{-12} \times 10^{-24} \text{cm}^2 = 1 \times 10^{-36} \text{cm}^2$
  - Integrated luminosity is expressed in units of inverse picobarns (pb$^{-1}$)
Colliding Beams

• Illustration at Fermilab (980 GeV protons on 980 GeV antiprotons)
  – Spring 2007, A good store had $<L> = 2 \times 10^{32} \text{ cm}^{-2}\text{s}^{-1}$
  – Achievement of 1 pb$^{-1}$ of integrated luminosity requires 5000 s (1.4 h) (Would see on average one event @ 1 pb cross section)
  – To accumulate 1 fb$^{-1}$ (10$^{-15}$ barns) requires 1400 hours.
Colliding Beams

• How to get the 2\textsuperscript{nd} beam?
  – Two rings of protons, ions, or electrons are used (e.g., RHIC, below), but expensive
  – Can use a single ring if one beam is antimatter (e.g., positrons or antiprotons) or if beams have opposite charges
Colliding Beams

• Answers
  – Stochastic cooling: store beam in a ring, apply feedback across ring diameter (or chord) to effect corrections
  – Electron cooling
    ✓ Use large mass difference between antiprotons and electrons
    ✓ Can travel together since charge the same
    ✓ Match velocities (make relativistic parameters $\beta$ and $\gamma$ equal)
    ✓ Transverse momentum transferred from antiprotons to electrons
    ✓ Electrons are refrigerant, take “heat” away, hence “cooling”

• Example at Fermilab:
  – $T = 8$ GeV ($\gamma = 9.53$) antiprotons already stochastically cooled
  – Cooled further using $T = 4.36$ MeV ($\gamma = 9.53$) electrons
  – Done in a unique ring of permanent magnets
  – Electrons come from a single-stage Pellatron
Colliding Beams

Fermilab Debuncher (left) and Accumulator (right) Rings
(Home of Stochastic Cooling)

Simon van der Meer, Photo from Nobel Prizes website.  Fermilab Rings, Photo from Fermilab website
Colliding Beams

Fermilab 8 GeV Recycler Ring (Permanent Magnets)
(Home of Electron Cooling)

G. Budker photo from Budker Institute website, other photos from Fermilab website
Synchrotrons and storage rings-radiation protection considerations

- All considerations pertaining to HV and RF Emittance (both longitudinal and transverse) and Courant-Synder parameters are important
- Gaussian functions may not adequately describe tails of beams
- Duty factor is a consideration
  - Instrument response questions
  - Safety interlock considerations
Synchrotrons and storage rings-radiation protection considerations

• Single-turn and resonant extraction pose different challenges
  – Single turn extraction may happen too fast for some corrective measures
  – Resonant extraction can, by design, produce large beams with increased beam loss

• Centering of beam is often important
  – To avoid scraping
  – To avoid quadrupole steering

• Long term storage of beam in storage rings
Synchrotrons and storage rings-radiation protection considerations

• Need to understand feasible and non-feasible beam losses
  – Can the entire beam be lost at a point?
  – Can this happen repeatedly?
  – Can the beam really be lost in a localized region in a circular machine if in a closed orbit?
    ✓ Injected, non-circulating beam is different.
    ✓ Extracted, non-circulating beam likewise is different.
    ✓ Time constants for magnets likely long compared with orbit periods
    ✓ Need input from accelerator physicists!
Future possibilities

- General techniques for accelerating/handing beams
  - Originated decades ago
  - Rely on macroscopic static & dynamic electromagnetic fields
  - Use special relativity, but have no direct connection with quantum world

- Phase velocity considerations do not allow free electromagnetic waves to accelerate particles
  - Energetic particles outrun the group velocity of waves
  - Accelerating $E$ fields in free waves are transverse to energy flow, not longitudinal
Future possibilities

• Use of lasers is tempting
  – Conservation of momentum & energy precludes transfer of all energy from individual photon to a particle.
  – But, can use photons collectively

• New development: laser-wakefield acceleration
  – Lasers are used to generate plasma waves
  – Waves drag particles along (like waves behind a boat)
  – $E$ fields in 10-100 GV m$^{-1}$ range.
Future possibilities

• New development: laser-wakefield acceleration
  – Difficulties
    ✓ Maintaining laser intensity over distances needed
    ✓ Large spreads in accelerated particle energies
    ✓ Slippage between velocity of relativistic particles ($\beta=1$) and velocity of the wake ($\beta<1$)
  – Breakthrough example at LBNL
    • Acceleration of 1.0 GeV electrons
    • In a distance of 0.033 m
  – There are other recent successes

• There are other techniques
Future possibilities

• Working at the quantum level
  – Interatomic electromagnetic and nuclear forces are much stronger than seen macroscopically.
  – Difficult to exploit
    ▶ Available only over short distances
    ▶ Often hidden as binding energy
  – Channeling by particles in a crystal (Fermilab example)
    • 900 GeV protons deflected by angle of 0.64 milliradians
    • Silicon crystal 0.039 m (long) X 0.003 m (high) X 0.009 m (wide), mechanically bent to achieve deflection
    • Bend achieved by same deflection in same length would need
      ◀ Static field of $E = 14.8 \text{ GV m}^{-1}$ (Eq. 17)
      ◀ Static field of $B = 49.3 \text{ T}$ (Eq. (20))
Conclusion

• Accelerator physicist needs to understand rudiments of operation of his/her particular installation

• Several read this material and offered comments
  – Dr. Michael Syphers - Fermilab accelerator physicist
  – Dr. Kamran Vaziri - Fermilab radiation physicist
  – Dr. Alex Elwyn - Fermilab radiation physicist (retired)

• Support was provided by
  – Mr. William Griffing, Fermilab ES&H Director, who encouraged my participation
  – My Fermilab co-workers who sat through a dry run.
  – My wife, Claudia, who allowed much work at home